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A note on uniqueness in the linearized water-wave problem

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The uniqueness theorem of Simon & Ursell (1984), concerning the linearized two-dimensional water-wave problem in a fluid of infinite depth, is extended in two directions. First, we consider a two-dimensional geometry involving two submerged symmetric bodies placed sufficiently far apart that they are not confined in the vertical right angle having its vertex on the free surface as the theorem of Simon & Ursell requires. A condition is obtained guaranteeing the uniqueness outside a finite number of bounded frequency intervals. Secondly, the method of Simon & Ursell is generalized to prove uniqueness in the axisymmetric problem for bodies violating John's condition provided the free surface is a connected plane region.

1. Introduction

In studies of water waves interacting with obstacles the question of uniqueness in the linearized problem is not yet fully answered despite its importance (see Ursell 1992, where this problem is placed first in the list of unfinished problems). The difficulty of the problem may be illustrated by the fact that during the forty years between 1950, when the pioneering papers by John (1950) and Ursell (1950) appeared, and 1990 only a dozen works were published on this topic (see the survey in McIver 1996).

In the last decade several results concerning the uniqueness and the existence of trapped modes (non-trivial solutions to the homogeneous boundary value problem, which lead to non-uniqueness in the non-homogeneous problem) have been obtained for obstacles separating a bounded portion of the free surface from infinity. The work in this direction was initiated by Kuznetsov (1988), who considered the two-dimensional problem for a pair of surface-piercing bodies subjected to rather strong geometrical restrictions. He demonstrated that the solution is unique for all frequencies below a value which depends on the geometry. This result was extended in Kuznetsov & Simon (1995*a*, *b*) and Simon & Kuznetsov (1996), where more two- and three-dimensional geometries were considered. However, all these papers contain restrictions on the single interval of frequencies in which uniqueness holds.

The reason for these restrictions became clear when McIver (1996) constructed the first example of a trapped mode for the two-dimensional problem. She applied the so-called inverse procedure which replaces seeking the eigenfrequencies for a given geometry by generating a two-body structure using an explicit potential. The latter involves two sources which do not radiate any waves to infinity (special spacing is chosen for this purpose) and has streamlines such that two of them can be interpreted as contours of surface-piercing bodies containing sources inside. The potential introduced by McIver (1996) would be deemed to be the lowest symmetric mode which may occur between the bodies. Kuznetsov & Porter (1999) considered higher symmetric as well as antisymmetric modes of the same type. These modes allow the construction of trapping structures consisting of more than two surface-piercing bodies. McIver & McIver (1997) constructed an example of non-uniqueness for the axisymmetric problem using the same method. Their result was extended by Kuznetsov & McIver (1997), who considered more examples of trapped modes arising in the presence of an axisymmetric surface-piercing toroidal body. These examples are in agreement with the theorem proven in the same paper and providing an infinite series of uniqueness intervals for all azimuthal modes. A similar theorem in two dimensions is published in Linton & Kuznetsov (1997). The latter theorem as well as examples of trapped modes from Kuznetsov & Porter (1999) are generalized to the case of oblique waves by Kuznetsov *et al.* (1998).

The uniqueness proofs in Linton & Kuznetsov (1997) and in Kuznetsov *et al.* (1998) rely essentially on two assumptions: (i) there are two surface-piercing bodies which are mirror reflections of each other in a vertical axis; (ii) John's condition must hold between the bodies. The first condition allows symmetric and antisymmetric solutions to be treated separately, and both conditions are used in a version of John's technique applied in the proofs. It involves a function having the form of an integral along vertical lines emanating from the inner part of the free surface, and the integrand depends on a solution to the homogeneous problem. Manipulations with this function allow an estimation of the potential energy between bodies from the kinetic energy strictly below the inner part of the free surface. Combining this estimate with that of Simon & Ursell (1984) for the exterior part of the free surface, one gets a contradiction proving that the solution is trivial. The first aim of the present paper is to show that more a sophisticated combination of the method developed in Linton & Kuznetsov (1997) with that of Simon & Ursell (1984) gives the possibility of extending the result of the latter paper, which states that the uniqueness theorem holds for an arbitrary number of arbitrary shaped bodies confined within a vertical right angle having its vertex on the free surface. Here we demonstrate the uniqueness theorem for a pair of submerged bodies symmetric about a vertical axis but having parts outside that angle. However, certain restrictions on the frequency must be imposed, but they give uniqueness intervals different from those in Kuznetsov & Simon (1995*a*, *b*).

The second result presented here is the generalization of Simon & Ursell's (1984) method to the axisymmetric water-wave problem for a body violating John's condition provided the free surface is a connected plane region.

The contents of the paper are as follows. In § 2 we prove the uniqueness result in two dimensions. Section 3 is devoted to the proof of uniqueness in the axisymmetric problem. The results obtained are discussed in § 4.

2. Uniqueness in the two-dimensional problem

The present section is concerned with the two-dimensional irrotational motion of an inviscid incompressible heavy fluid. Surface tension effects are neglected. The motion is assumed to be time-harmonic and of small amplitude. Thus, a velocity potential exists and can be written in the form $\mathrm{Re}\left\{\Phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t}\right\}$. Here (x,y) are rectangular Cartesian coordinates with origin in the mean free surface, and with the y-axis directed upwards.

We denote by S_+ a piecewise smooth contour submerged in deep water so that any

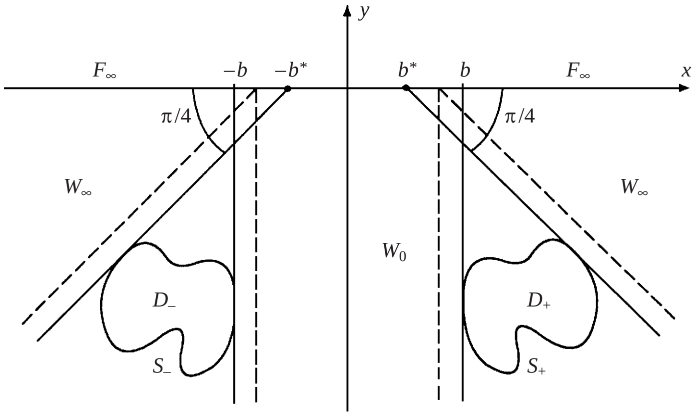

FIGURE 1. A defintion sketch of the geometry.

vertical line on the left of $x = b$ $(b > 0)$ does not intersect S_+, and a vertical line $x = c$ intersects this contour for any c, belonging to a certain finite interval with the left end-point b. Furthermore, we assume that S_+ lies below $y = -x + b_0$, where $0 < b_0 \leqslant b$ (the case $b_0 = 0$ was considered in Simon & Ursell 1984). We put $b^* = \min b_0$, where the minimum is taken over all b_0, such that $y = -x + b_0$ is above S_+. In order to generalize the theorem proved by Simon & Ursell (1984) we have to assume that $b^* > 0$. Let S_- be the reflection of S_+ by the y-axis. We denote by W the whole fluid domain, that is, $W = \mathbb{R}^2_- \backslash (D_+ \cup D_-)$, where $\mathbb{R}^2_- = \{-\infty < x < +\infty,\ y < 0\}$, and $D_\pm$ is enclosed in $S_\pm$. Let W_0 be the semistrip $\{|x| < b_0,\ -\infty < y < 0\}$, and W_∞ be the union of two parts of W: one lying to the right of $y = -x + b_0$, and the other one to the left of $y = x - b_0$. The geometry described is shown in figure 1, where the boundaries of W_0 and W_∞ are shown by dashed lines.

The two problems arising in applications of the radiation and scattering of waves by a set of rigid bodies are formulated as a linear boundary value problem in which $\partial\Phi/\partial n$ is prescribed on the wetted surface of the bodies. Then the question of uniqueness reduces to a demonstration that the difference $u = \Phi_1 - \Phi_2$ of two solutions vanishes. In the case under consideration the function u (it is assumed to be real, because otherwise its real and imaginary parts must be considered separetely) must satisfy the following homogeneous equation and boundary conditions:

$$\nabla^2 u = 0 \quad \text{in} \quad W, \tag{2.1}$$

$$u_y - \nu u = 0 \quad \text{on} \quad F, \tag{2.2}$$

$$\partial u/\partial n = 0 \quad \text{on} \quad S. \tag{2.3}$$

Here $\nu = \omega^2/g$, g is the acceleration due to gravity, F denotes a free surface $\{-\infty < x < +\infty,\ y = 0\}$, and $S = S_+ \cup S_-$.

The problem (2.1)–(2.3) must be complemented by the radiation condition. However, it is well-known (see, for example, Simon & Ursell 1984) that a solution to the homogeneous water-wave problem satisfies the following condition:

$$\int_W |\nabla u|^2 \,\mathrm{d}x\mathrm{d}y + \nu \int_F |u|^2 \,\mathrm{d}x < \infty, \tag{2.4}$$

which means that the kinetic and potential energy are finite. Moreover, Green's formula gives for u:

$$\int_W |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y - v \int_F |u|^2 \, \mathrm{d}x = 0. \tag{2.5}$$

Using the symmetry of the water domain W we can decompose u into the sum of a symmetric part $u^{(+)}$, and an antisymmetric part $u^{(-)}$ defined as follows:

$$u^{(\pm)}(x, y) = \pm u^{(\pm)}(-x, y).$$

It is obvious that

$$u_x^{(+)}(0, y) = 0, \quad u^{(-)}(0, y) = 0. \tag{2.6}$$

The considerations in § 5 of Simon & Ursell (1984) (in particular, see equations (5.10)–(5.12) on p. 146 in their paper) imply that:

$$v \int_{F_\infty} |u^{(\pm)}|^2 \, \mathrm{d}x < \int_{W_\infty} |\nabla u^{(\pm)}|^2 \, \mathrm{d}x\mathrm{d}y, \tag{2.7}$$

where $F_\infty = \{|x| > b_0, \ y = 0\}$.

As in Linton & Kuznetsov (1997) we introduce

$$w^{(\pm)}(x) = \int_{-\infty}^0 u^{(\pm)}(x, y)\mathrm{e}^{vy} \, \mathrm{d}y$$

on the part of the free surface $\{-b < x < b, \ y = 0\}$. Using integration by parts and the free surface condition (2.2), it is shown in Linton & Kuznetsov (1997) that

$$w_{xx}^{(\pm)} + v^2 w^{(\pm)} = 0 \quad \text{for} \quad -b < x < b.$$

Then (2.6) implies that

$$w^{(\pm)}(x) = C_\pm \cos\left(vx - \frac{\pi}{4} \pm \frac{\pi}{4}\right),$$

where $C_\pm$ are real constants. Integration by parts in

$$C_\pm \cos\left(vx - \frac{\pi}{4} \pm \frac{\pi}{4}\right) = \int_{-\infty}^0 u^{(\pm)}(x, y)\mathrm{e}^{vy} \, \mathrm{d}y \tag{2.8}$$

leads to

$$u^{(\pm)}(x, 0) = vC_\pm \cos\left(vx - \frac{\pi}{4} \pm \frac{\pi}{4}\right) + \int_{-\infty}^0 u_y^{(\pm)}(x, y)\mathrm{e}^{vy} \, \mathrm{d}y,$$

from which

$$|u^{(\pm)}(x, 0)|^2 \leqslant 2\left[v^2 C_\pm^2 \cos^2\left(vx - \frac{\pi}{4} \pm \frac{\pi}{4}\right) + \left(\int_{-\infty}^0 u_y^{(\pm)}(x, y)\mathrm{e}^{vy} \, \mathrm{d}y\right)^2\right].$$

Applying the Schwarz inequality to the last integral we get

$$v|u^{(\pm)}(x, 0)|^2 \leqslant 2v^3 C_\pm^2 \cos^2\left(vx - \frac{\pi}{4} \pm \frac{\pi}{4}\right) + \int_{-\infty}^0 \left|u_y^{(\pm)}(x, y)\right|^2 \, \mathrm{d}y. \tag{2.9}$$

On the other hand, we have from (2.8),

$$-vC_\pm \sin\left(vx - \frac{\pi}{4} \pm \frac{\pi}{4}\right) = \int_{-\infty}^0 u_x^{(\pm)}(x, y)\mathrm{e}^{vy} \, \mathrm{d}y,$$

which implies that

$$2\nu^3 C_{\pm}^2 \sin^2\left(\nu x - \frac{\pi}{4} \pm \frac{\pi}{4}\right) \leqslant \int_{-\infty}^{0} \left|u_x^{(\pm)}(x,y)\right|^2 \,\mathrm{d}y. \tag{2.10}$$

Let us assume that

$$\pi\left(m + \tfrac{1}{4} \pm \tfrac{1}{4}\right) \leqslant \nu b_0 \leqslant \pi\left(m + \tfrac{3}{4} \pm \tfrac{1}{4}\right), \quad m = 0, 1, \ldots, \tag{2.11}$$

which is equivalent to

$$\int_0^{b_0} \left[\cos^2\left(\nu x - \frac{\pi}{4} \pm \frac{\pi}{4}\right) - \sin^2\left(\nu x - \frac{\pi}{4} \pm \frac{\pi}{4}\right)\right] \mathrm{d}x \leqslant 0.$$

Then integrating (2.9) and (2.10) over $F_0 = \{-b_0 < x < b_0,\ y = 0\}$ under assumption (2.11), we arrive at

$$\nu \int_{F_0} \left|u^{(\pm)}(x,0)\right|^2 \mathrm{d}x \leqslant \int_{W_0} \left|\nabla u^{(\pm)}\right|^2 \mathrm{d}x\mathrm{d}y.$$

Adding this inequality to (2.7) produces

$$\nu \int_{F} \left|u^{(\pm)}(x,0)\right|^2 \mathrm{d}x \leqslant \int_{W_0 \cup W_\infty} \left|\nabla u^{(\pm)}\right|^2 \mathrm{d}x\mathrm{d}y.$$

This contradicts (2.5), which is true for $u^{(\pm)}$, unless $u^{(\pm)} \equiv 0$ in W. Thus, *the symmetric* (*antisymmetric*) *solution is unique when* ν *satisfies* (2.11)*, where the sign* $+$ ($-$) *is taken, and* b_0 *belongs to* $[b^*, b]$.

Dividing (2.11) by b_0, we see that putting $b_0 = b$ we get the best lower bound for ν, equal to $\pi b^{-1}\left(m + \frac{1}{4} \pm \frac{1}{4}\right)$. Similarly, putting $b_0 = b^*$ we get the best upper bound for ν, equal to $\pi b^{*-1}\left(m + \frac{3}{4} \pm \frac{1}{4}\right)$. Hence, *the uniqueness is guaranteed for the symmetric* (*antisymmetric*) *solution when the inequality*

$$\pi(m + \tfrac{1}{4} \pm \tfrac{1}{4}) \leqslant \nu b \leqslant \pi(m + \tfrac{3}{4} \pm \tfrac{1}{4})\frac{b}{b^*} \tag{2.12}$$

holds with $+$ ($-$)*, where* $m = 0, 1, \ldots$.

If $b/b^* > 1$, then the above assertion implies that non-uniqueness might occur only for ν belonging to a finite number of intervals. From (2.12) we see that if m_+ is the smallest non-negative integer, such that

$$(m_+ + 1)\frac{b}{b^*} \geqslant m_+ + \tfrac{3}{2}, \tag{2.13}$$

then the right-hand end of the uniqueness interval (2.12) with $m = m_+$ for the symmetric solutions belongs to the similar interval with $m = m_+ + 1$. Hence, *if* m_+ *is the smallest non-negative integer satisfying* (2.13)*, then the non-uniqueness of symmetric modes might occur only for* ν *belonging to the first* $m_+ + 1$ *intervals defined by* (2.12) *with the sign* $-$*. In particular, when* $b \geqslant 3b^*/2$*, the symmetric solution is unique for all* $\nu > 0$ *with the possible exception of the interval* $(0, \pi(2b)^{-1})$.

Similarly, *if* m_- *is the smallest non-negative integer satisfying*

$$(m_- + \tfrac{1}{2})\frac{b}{b^*} \geqslant m_- + 1,$$

then the non-uniqueness of antisymmetric modes might occur only for ν *belonging to the first* m_- *intervals defined by* (2.12) *with the sign* $+$*. In particular, when* $b \geqslant 2b^*$*, the antisymmetric solution is unique for all* $\nu > 0$.

3. The axisymmetric problem in fluid of infinite depth

The method developed in Simon & Ursell (1984) relies on the use of the Cauchy–Riemann type relation (see (5.3) in their paper). Since there are similar relations in the axisymmetric case, one may expect the same idea to work in the axisymmetric problem. Thus, we assume W to be an axisymmetric fluid domain, having no cusps and zero-angled edges on ∂W, which consists of the free surface $F = \partial W \cap \{y = 0\}$, and of an axisymmetric wetted rigid surface S. We assume that F is a connected plane region, and a point x in this plane has coordinates (x_1, x_2), so that $|x|^2 = x_1^2 + x_2^2$.

Let the velocity potential of the form $u(|x|, y)$ be a solution to the homogeneous problem (2.1)–(2.3). We assume u to be real without loss of generality. Since there are no cusps and zero-angled edges on ∂W, the general results from the theory of elliptic boundary value problems in piecewise smooth domains (see Kondrat'yev 1967; Kozlov, Maz'ya & Rossmann 1997; and Nazarov & Plamenevsky 1994) yield that the kinetic and potential energy of waves defined by u are locally finite in W. Then, (2.4) holds, and as in § 2 we arrive at (2.5). Our aim is to derive an inequality contradicting (2.5), which implies that the uniqueness theorem holds for the problem.

Consider the conical surface S_d generated by revolving the line

$$\ell_d(\beta) = \{(x, y) :\ y = (d - x_1)\tan\beta,\ x_1 > d,\ x_2 = 0,\ y < 0\} \tag{3.1}$$

around the y-axis, so that S_d has waterline intersection $|x| = d$, and the dihedral angle between S_d and $\{|x| > d,\ y = 0\}$ is equal to $\beta \in (0, \pi/2]$. Provided any bodies are inside S_d, we have

$$0 = \int_{S_d} \left(u\frac{\partial \phi}{\partial n} - \phi\frac{\partial u}{\partial n} \right) \mathrm{d}S = 2\pi \int_{\ell_d} \left(u\frac{\partial \phi}{\partial n} - \phi\frac{\partial u}{\partial n} \right) |x|\,\mathrm{d}s. \tag{3.2}$$

Here $\phi(|x|, y)$ is an axisymmetric harmonic function in the water domain, which satisfies the free surface boundary condition and is bounded as $|x|^2 + y^2 \to \infty$. Let ψ be related to ϕ through the following equations:

$$\frac{\partial \phi}{\partial |x|} = -\frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = |x|^{-1}\frac{\partial(|x|\psi)}{\partial |x|}, \tag{3.3}$$

which are similar to relations between the velocity potential and stream function in the axisymmetric case. These equations lead to

$$|x|\frac{\partial \phi}{\partial n} = -\frac{\partial(|x|\psi)}{\partial s} \quad \text{on} \quad \ell_d,$$

where $\boldsymbol{n}$ and $\boldsymbol{s}$ are defined as follows. The vector $\boldsymbol{s}$ is directed along ℓ_d from infinity to the plane $\{y = 0\}$, and $(\boldsymbol{n},\ \boldsymbol{s})$ form a right-hand pair of vectors. Hence, we get from (3.2) that

$$\begin{aligned}\int_{\ell_d} \phi\frac{\partial u}{\partial n}|x|\,\mathrm{d}s = \int_{\ell_d} u\frac{\partial \phi}{\partial n}|x|\,\mathrm{d}s = -\int_{\ell_d} u\frac{\partial(|x|\psi)}{\partial s}\,\mathrm{d}s \\ = -du(d,0)\psi(d,0) + \int_{\ell_d} \psi\frac{\partial u}{\partial s}|x|\,\mathrm{d}s.\end{aligned}$$

As in Simon & Ursell (1984), use of ψ has allowed an integration by parts. The result is that $u(d, 0)$ is expressed as an integral along ℓ_d; specifically,

$$du(d,0)\psi(d,0) = \int_{\ell_d} \left(\phi\frac{\partial u}{\partial n} - \psi\frac{\partial u}{\partial s} \right) |x|\,\mathrm{d}s. \tag{3.4}$$

Now, $F = \{|x| > d_{\min},\ y = 0\}$ for some $d_{\min}$ which will be zero if all bodies are fully submerged, but which will be non-zero if there is a body intersecting the free surface. We wish to bound

$$v \int_F |u|^2 \,\mathrm{d}x = 2\pi v \int_{d_{\min}}^{\infty} |u(d,0)|^2 d \,\mathrm{d}d$$

in terms of $\int_{W_c} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}y$, where $W_c \subset W$, and is swept out by conical surfaces of the family $\{S_d\}$. By (3.4) we must bound

$$v d|u(d,0)|^2 = v \frac{\left| \int_{\ell_d} \left(\phi \frac{\partial u}{\partial n} - \psi \frac{\partial u}{\partial s} \right) |x| \,\mathrm{d}s \right|^2}{d|\psi(d,0)|^2} \tag{3.5}$$

in terms of $\int_{\ell_d} |\nabla u|^2 |x| \,\mathrm{d}y$. It is appropriate to put

$$\phi(|x|, y) = \mathrm{e}^{vy} H_0^{(1)}(v|x|) \quad \text{and} \quad \psi(|x|, y) = \mathrm{e}^{vy} H_1^{(1)}(v|x|),$$

where $H_0^{(1)}$ and $H_1^{(1)}$ are Hankel functions. The formulae 9.1.30 in Abramowitz & Stegun (1964) guarantee that ϕ and ψ satisfy (3.3). Then, taking into account (3.1) we obtain

$$\frac{1}{\sin^2 \beta} \left| \int_{\ell_d} \left[\frac{H_0^{(1)}(v|x|)}{H_1^{(1)}(v|x|)} \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} \right] \mathrm{e}^{vy} \frac{|x|^{1/2} H_1^{(1)}(v|x|)}{d^{1/2} H_1^{(1)}(vd)} |x|^{1/2} \,\mathrm{d}y \right|^2 \tag{3.6}$$

for the left-hand side of (3.5).

It is known that $k \left| H_1^{(1)}(k) \right|^2$ is a monotonically decreasing function (see Gradshteyn & Ryzhik 1980, 8.478).

This assertion and the Schwarz inequality imply that

$$v d|u(d,0)|^2 \leqslant \frac{1}{2\sin^2 \beta} \int_{\ell_d} \left| \frac{H_0^{(1)}(v|x|)}{H_1^{(1)}(v|x|)} \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} \right|^2 |x| \,\mathrm{d}y. \tag{3.7}$$

This means that we have to estimate

$$\left| \frac{H_0^{(1)}(v|x|)}{H_1^{(1)}(v|x|)} \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} \right|^2 \left| \frac{\partial u}{\partial s} + i \frac{\partial u}{\partial n} \right|^{-2}, \tag{3.8}$$

which is equivalent to finding

$$\sup_{X \in \mathbb{R}} \frac{|X + p + \mathrm{i}q|^2}{|X + \mathrm{i}|^2} = 1 + \sup_{X \in \mathbb{R}} \frac{2pX + c}{X^2 + 1}, \tag{3.9}$$

where

$$\frac{H_0^{(1)}(v|x|)}{H_1^{(1)}(v|x|)} = p + \mathrm{i}q \quad \text{and} \quad c = p^2 + q^2 - 1. \tag{3.10}$$

It is obvious that $p = [J_0 J_1 + Y_0 Y_1]/|H_1^{(1)}|^2$, and

$$q = [J_1 Y_0 - J_0 Y_1] / \left| H_1^{(1)} \right|^2 = 2 / \left\{ \pi v |x| \left| H_1^{(1)}(v|x|) \right|^2 \right\}. \tag{3.11}$$

The last equality is a consequence of 9.1.16 in Abramowitz & Stegun (1964) for Wronskian of Bessel functions, and hence, $0 < q < 1$.

The maximum in (3.9) occurs at X_m such that

$$2pX_m + c = (c^2 + 4p^2)^{1/2} \quad \text{and} \quad X_m^2 + 1 = 2 - \frac{c}{p} X_m = (c^2 + 4p^2)^{1/2} \frac{X_m}{p},$$

and so

$$\sup_{X \in \mathbb{R}} \frac{2pX + c}{X^2 + 1} = \frac{c + (c^2 + 4p^2)^{1/2}}{2}.$$

Substituting c from (3.10), we find that the maximum in (3.9) is equal to

$$1 + \tfrac{1}{2}\left\{p^2 + q^2 - 1 + \left[(p^2 + q^2 - 1)^2 + 4p^2\right]^{1/2}\right\} \\ = \tfrac{1}{2}\left\{p^2 + q^2 + 1 + \left[(p^2 + q^2 + 1)^2 - 4q^2\right]^{1/2}\right\}.$$

Substituting $p^2 + q^2$ from (3.10) and q from (3.11) into the above expression we get a bound for (3.8), which together with (3.7) gives the following inequality:

$$\nu d |u(d, 0)|^2 \leqslant \frac{M}{2 \sin^2 \beta} \int_{\ell_d} |\nabla u|^2 |x| \, \mathrm{d}y,$$

where

$$M = \tfrac{1}{2} \sup_{X \in \mathbb{R}} \left|H_1^{(1)}(X)\right|^{-2} \left\{ \left|H_0^{(1)}(X)\right|^2 + \left|H_1^{(1)}(X)\right|^2 \right. \\ \left. + \left(\left[\left|H_0^{(1)}(X)\right|^2 + \left|H_1^{(1)}(X)\right|^2\right]^2 - \frac{16}{(\pi X)^2} \right)^{1/2} \right\}. \quad (3.12)$$

This inequality implies that

$$\nu \int_F |u|^2 \, \mathrm{d}x \leqslant \frac{M}{2 \sin^2 \beta} \int_{W_c} |\nabla u|^2 \, \mathrm{d}x \mathrm{d}y.$$

If $M/(2 \sin^2 \beta) \leqslant 1$, then this contradicts (2.5) unless u vanishes identically in W. Thus, the following uniqueness theorem is proved.

Let W be an axisymmetric water domain, such that the free surface F is a connected plane region. Let any cone S_d, obtained by rotation of the line (3.1) *about the y-axis, belong to W for $\{|x| = d, \ y = 0\} \in F$, and let $M/(2 \sin^2 \beta) \leqslant 1$, where M is defined by* (3.12). *Then the homogenous axisymmetric water-wave problem has only a trivial solution.*

It is obvious that the inequality $M < 2 \sin^2 \beta$ is true for $\beta = \pi/2$, because in this case the theorem of John (1950) guarantees the uniqueness. The constant M can be evaluated numerically from (3.12), which gives $M \approx 1.2$, and this maximum occurrs at $X \approx 0.8$. Hence, there exists a β_0 such that $0.6 \approx \sin^2 \beta_0$, that is, $\beta_0 \approx 52°$, and if $\beta > \beta_0$, then the uniqueness theorem holds. More thorough calculation shows that uniqueness is available if $\beta > 52°$.

4. Discussion

Uniqueness has been established in the two-dimensional and axisymmetric water-wave problems for infinite fluid depth, under different geometrical restrictions generalizing those in Simon & Ursell (1984). Here we present a simple analysis of the uniqueness conditions.

We consider two-dimensional theorems first. Let S_+ satisfy the assumptions of theorems proven in § 2 with certain values of b and b_*. If the depth of submergence of bodies increases whilst the spacing $2b$ remains the same, then b_* decreases. Then, the number of the frequency intervals of possible non-uniqueness becomes smaller, and for sufficiently large depth of submergence the uniqueness holds for all frquencies (when $b_* = 0$). If the depth of submergence decreases, then the number of interals of possible non-uniqueness increases and becomes infinite for $b_* = b$, when the theorem has the same form as in the case of surface-piercing pairs of symmetric bodies (cf. Linton & Kuznetsov 1997).

When the depth of submergence remains constant and spacing decreases, we have the same situation as in the case of increasing depth of submergence. However, the smallest number of intervals of possible non-uniqueness might not be zero and depends on the geometry. On increasing the spacing we increase b and b_* simultaneously, leaving $b - b_* = \text{const}$. Then, it follows from (2.13) that the number of intervals of possible non-uniqueness increases.

To illustrate the uniqueness theorem for the axisymmetric problem we consider an immersed sphere of radius a. If it is immersed less than half or half-immersed, then uniqueness follows from John's theorem. Let the centre of the sphere be placed at $y = -h$, $h > 0$. Then, the theorem in § 3 provides uniquess for a more than half-immersed sphere, when either

$$h \leqslant a \cos \beta_0 \quad \text{or} \quad a \leqslant h \cos \beta_0,$$

where β_0 is defined in the end of § 3. In the former case the sphere is immersed partially, and in the latter case it is immersed totally. In fact, any totally submerged sphere is known to have the uniqueness property in the axisymmetric water-wave problem as has been shown by Livshits (1974).

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